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Two-Body Problem with Drag and High Tangential Speeds

Thomas Carter*
Eastern Connecticut State University, Willimantic, Connecticut 06226
and
Mayer Humi†
Worcester Polytechnic Institute, Worcester, Massachusetts 01609

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This paper considers the restricted two-body problem with atmospheric drag. A simple formula is presented that approximates the atmospheric density from raw data, replaces previous models, and is amenable to closed-form solution of the orbit equation for high tangential speeds. A procedure for subdividing an altitude interval and calculating the parameters of the formula over each subinterval leads to highly improved accuracy in the solutions and compares favorably with numerical integration. To validate our model we compare it with the trajectory and flight time of a satellite in an exponential atmosphere starting from a near-circular orbit at 7120 km from the Earth's center.

I. Introduction

THE problem of finding closed-form solutions of the restricted two-body problem with atmospheric drag is interesting from both an academic and practical point of view. Some early work on this problem is summarized in the paper by Lane and Cranford [1] using perturbations and orbital elements. Later results in this area are represented by the works of Hoots and France [2], King-Hele [3] and more recently Vallado [4]. Recently Humi and Carter [5–9] have approximated the equations of motion for the very important class of problems in which the radial speeds are relatively small in comparison with the tangential speeds, and have found solutions under these assumptions. The present paper completes this approach attempting to unify and extend some of these results.

The equation of motion of a satellite about a spherical planet under the influence of Newtonian gravitation and atmospheric drag is assumed [7] to be of the form

$$\ddot{\mathbf{R}} = -f(R)\mathbf{R} - \alpha\rho(R)(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}})^{1/2}\dot{\mathbf{R}} \tag{1}$$

where $\mathbf{R}$ represents the position vector of the satellite from the center of attraction, $R = (\mathbf{R}\cdot\mathbf{R})^{1/2}$ is the magnitude of this vector in which the dot indicates the scalar product, but the upper dots represents differentiation with respect to time t. The gravitational function is $f(R) = \mu/R^3$ where μ is the product of the universal gravitational constant and the mass of the planet, $\rho(R)$ is directly proportional to the atmospheric density at a distance R from the center of attraction, and α is a constant that is determined from the drag coefficient of the satellite, its geometry, and the atmospheric density at a specified altitude

It has been shown [7] that the motion defined by Eq. (1) is planar, and can be written in terms of polar coordinates R, θ in this plane as follows:

$$R\ddot{\theta} + 2\dot{R}\,\dot{\theta} = -\,\alpha\rho(R)(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}})^{1/2}R\dot{\theta} \tag{2}$$

$$\ddot{R} - R\dot{\theta}^2 = -f(R)R - \alpha\rho(R)(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}})^{1/2}\dot{R} \tag{3}$$

Dividing Eq. (2) by $R\dot{\theta}$, and integrating we obtain the instantaneous specific angular momentum

$$J = R^2\dot{\theta} = he^{-\alpha\int\rho(R)(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}})^{1/2}\,\mathrm{d}t} \tag{4}$$

where h is a constant of integration that represents its initial value.

We use Eq. (4) to change the independent variable from t to θ and make this change in Eq. (3). After some manipulations we obtain the orbit equation

$$RR''(\theta) - 2R'(\theta)^2 = R^2 - f(R)R^6/J^2 \tag{5}$$

where the prime indicates differentiation with respect to θ, and J is now regarded as a function of θ.

In accordance with previous work [5–9] we consider only motion where $|\dot{R}| \ll |R\dot{\theta}|$ as in the case of arcs that are nearly circular. This simplifies the formula for the instantaneous specific angular momentum

$$J = he^{-\alpha\int\rho(R)R\,\mathrm{d}\theta} \tag{6}$$

The fundamental problem of the preceding papers [5–8] and of this present paper is to approximate the density function $\rho(R)$ with reasonable accuracy in such a way as to devise closed-form solutions of Eqs. (5) and (6). A typical model for the density of the Earth's atmosphere at heights near 7000 km above the center of the Earth has been given by

$$\rho_{\mathrm{exp}} = \rho_0 e^{-\frac{(R-R_0)}{H}} \tag{7}$$

where $H = 88.667$ km and $R_0 = 7120$ km. Some success, at least locally, has been found by replacing Eq. (7) by the following:

$$\rho(R) = \frac{A}{R} \tag{8}$$

$$\rho(R) = \frac{A}{R^2} \tag{9}$$

$$\rho(R) = \frac{A}{R-c} \tag{10}$$

where A and c are constants. The model presented here is an improvement over these models.

The approximations (8) and (9) led to concise solutions, but become increasingly inaccurate as R is varied more than a few kilometers [5,7]. The approximation (10) was much more accurate

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*Professor Emeritus, Department of Mathematics; cartert@easternct.edu. Senior Member AIAA.

†Professor, Department of Mathematical Sciences; mhumi@wpi.edu.

over a wider range of R, but the form of the solution was less concise [8].

In the present paper, we present a formula for $\rho(R)$ that includes all of the above as special cases, and the parameters c and δ are replaced with more natural parameters that are determined herein. This new approach does not require that the atmospheric density be modeled by Eq. (7), but we find an exponential model convenient to check the accuracy of our formulas using computer simulations. Actually, this approach can be used even if the atmospheric density is defined only from raw data

$$\rho(R_1), \rho(R_2), \ldots, \rho(R_k) \tag{11}$$

This work can be applied in two distinct ways. If the data $R_1, \ldots, R_k$ are reasonably close, we present a formula for $\rho(R)$ that provides a highly accurate approximation. Contrariwise, if the data $R_1, \ldots, R_k$ are more numerous and widely spread, we present a sequence of functions $\{\rho_i(R)\}$ that accurately approximate the data in a more global sense. The accuracy of these applications is verified by computer simulations.

II. New Formulas for Radial Distance

This paper presents the following approximate atmospheric density function:

$$\rho(R) = \frac{k_1}{R} + \frac{k_2}{R^2} \tag{12}$$

where the constants k_1, k_2 are selected to fit the data, Eq. (11). Clearly this function is general enough to encompass Eqs. (8) and (9) by, respectively, setting k_2 or k_1 to zero. Moreover this function is an improvement over Eq. (10) because it avoids the approximation presented in Eq. 17 of [8] and the resulting selection of the small positive number δ. The analysis presented here is an improvement over that of [8].

A. Radial Distance *R*

Inserting Eq. (12) in Eq. (6) we obtain

$$J = he^{-\alpha[k_1(\theta-\theta_1)+k_2 u]} \tag{13}$$

where $\theta_1 = \theta(t_1)$ at the initial time t_1 and

$$u = \int_{\theta_1}^{\theta} \frac{\mathrm{d}\phi}{R(\phi)} \tag{14}$$

Differentiating Eq. (14), substituting into the orbit Eq. (5), and simplifying we obtain

$$u''' + u' = \frac{\mu}{h^2} e^{2\alpha[k_1(\theta-\theta_1)+k_2 u]} \tag{15}$$

Since $\alpha \ll 1$, we may linearize the right-hand side, resulting in the linear differential equation

$$u''' + u' - \frac{2\alpha\mu}{h^2} k_2 u = \frac{\mu}{h^2} + \frac{2\alpha\mu}{h^2} k_1(\theta - \theta_1) \tag{16}$$

The characteristic equation associated with the homogeneous part of this differential equation is

$$\lambda^3 + \lambda - \frac{2\alpha\mu k_2}{h^2} = 0 \tag{17}$$

which has one positive root

$$\lambda = a > 0 \tag{18}$$

and two complex roots

$$\lambda = \sigma \pm \omega i \tag{19}$$

Solving the cubic Eq. (17) we find that

$$a = \beta - \frac{1}{3\beta} \tag{20}$$

where

$$\beta = \left\{ \frac{\mu\alpha k_2}{h^2} + \frac{[1 + 27(\frac{\mu\alpha k_2}{h^2})^2]^{1/2}}{3\sqrt{3}} \right\}^{1/3} \tag{21}$$

$$\sigma = -a/2 \tag{22}$$

and

$$\omega = \left(1 + \frac{3a^2}{4}\right)^{1/2} \tag{23}$$

If $k_2 \neq 0$, the complete solution of Eq. (16) can be expressed in terms of these parameters and the arbitrary constants c_1, c_2, and ϕ as follows:

$$u(\theta) = c_1 e^{a\theta} + c_2 e^{-\frac{a\theta}{2}} \cos\omega(\theta - \theta_1) - \frac{h^2 k_1}{2\alpha\mu k_2^2} - \frac{1}{2\alpha k_2} - \frac{k_1(\theta - \theta_1)}{k_2} \tag{24}$$

If $k_2 = 0$ the solution is simpler and is presented in [5].

It is convenient to define new parameters ν, ψ, and θ_0 by

$$\nu = \left(\omega^2 + \frac{a^2}{4}\right)^{1/2} \tag{25}$$

$$\sin\omega\psi = \frac{\omega}{\nu}, \qquad \cos\omega\psi = \frac{a}{2\nu} \tag{26}$$

$$\theta_0 = \phi + \psi \tag{27}$$

We differentiate Eq. (24) and express it in terms of these parameters, obtaining

$$u'(\theta) = ac_1 e^{a\theta} - \nu c_2 e^{-\frac{a\theta}{2}} \cos\omega(\theta - \theta_0) - \frac{k_1}{k_2} \tag{28}$$

If c_1 and k_2 are not zero, then this expression and Eq. (14) produce the complete solution of the orbit of Eq. (5) for the atmospheric density Eq. (12) under the assumption that $|\dot{R}| \ll |R\dot{\theta}|$. Since $u' = 1/R(\theta)$, we can solve this expression for $R(\theta)$, and divide numerator and denominator by $ac_1 e^{a\theta} - k_1/k_2$. We are seeking a solution that generalizes the semiparameter P and eccentricity ϵ in the solution of the two-body problem without drag. We find these respective generalizations to be as follows:

$$P(\theta) = \frac{k_2 e^{-a\theta}}{k_2 ac_1 - k_1 e^{-a\theta}}, \qquad \epsilon(\theta) = |c_2|\nu P(\theta) e^{-a\theta/2} \tag{29}$$

but the generalized two-body solution for the radial distance takes three forms: If $c_2 > 0$, then

$$R(\theta) = \frac{P(\theta)}{1 - \epsilon(\theta)\cos\omega(\theta - \theta_0)} \tag{30}$$

but if $c_2 < 0$, then

$$R(\theta) = \frac{P(\theta)}{1 + \epsilon(\theta)\cos\omega(\theta - \theta_0)} \tag{31}$$

Finally, if $c_2 = 0$, the solution degenerates into a spiral

$$R(\theta) = P(\theta) \tag{32}$$

B. Evaluation of the Arbitrary Constants

We now regard time as the independent variable.

To determine the constants c_1, c_2, θ_0 that appear in Eqs. (29–31) we use the initial data $\theta(t_1)$, $R(t_1)$, $\dot{R}(t_1)$ and $R^2(t_1)\dot{\theta}(t_1)$ associated with the initial time t_1. For convenience we rewrite $\theta(t_1) = \theta_1$, $R(t_1) = R_1$, $\dot{R}(t_1) = v_1$, and $R^2(t_1)\dot{\theta}(t_1) = h$.

Since $u'(\theta) = 1/R$, we have

$$\frac{1}{R} = -\frac{k_1}{k_2} + ac_1 e^{a\theta} - \nu c_2 e^{-a/2\theta} \cos\omega(\theta - \theta_0) \tag{33}$$

Differentiating this equation with respect to t and using the fact that $(\mathrm{d}R/\mathrm{d}t = \mathrm{d}R/\mathrm{d}\theta)\dot{\theta}$ we have

$$-\frac{\dot{R}}{R^2\dot{\theta}} = a^2 c_1 e^{a\theta} + \nu^2 c_2 e^{-a/2\theta} \cos\omega(\theta - \theta_0 - \psi) \tag{34}$$

Finally, a third equation is obtained from Eq. (14) by setting $u(\theta_1)$ equal to zero. Using Eq. (24) this initial condition leads to

$$C_1 + C_2 \cos\omega(\theta_1 - \theta_0 + \psi) = S_1 \tag{35}$$

similarly, Eqs. (33) and (34) at time t_1 yield

$$aC_1 - \nu C_2 \cos\omega(\theta_1 - \theta_0) = S_2 \tag{36}$$

$$a^2 C_1 + \nu^2 C_2 \cos\omega(\theta_1 - \theta_0 - \psi) = S_3 \tag{37}$$

These equations have been simplified by the substitutions

$$C_1 = c_1 e^{a\theta_1}, \qquad C_2 = c_2 e^{-a/2\theta_1} \tag{38}$$

The right-hand sides are defined by

$$S_1 = \frac{1}{2\alpha k_2}\left(1 + \frac{h^2 k_1}{\mu k_2}\right) \tag{39}$$

$$S_2 = \frac{1}{R_1} + \frac{k_1}{k_2} \tag{40}$$

$$S_3 = -\frac{v_1}{h} \tag{41}$$

Solving Eqs. (36) and (37) for C_1 and C_2 we obtain

$$C_1 = \frac{(aS_2 + 2S_3)\cos\omega(\theta_1 - \theta_0) + 2\omega S_2 \sin\omega(\theta_1 - \theta_0)}{3a^2 \cos\omega(\theta_1 - \theta_0) + 2a\omega \sin\omega(\theta_1 - \theta_0)} \tag{42}$$

$$C_2 = \frac{-2aS_2 + 2S_3}{3a\nu \cos\omega(\theta_1 - \theta_0) + 2\omega\nu \sin\omega(\theta_1 - \theta_0)} \tag{43}$$

We can now solve for θ_0 by substituting these into Eq. (35). We obtain

$$\tan\omega(\theta_1 - \theta_0) = \frac{3a^2\nu^2 S_1 + a(a^2 - \nu^2)S_2 - (2\nu^2 + a^2)S_3}{2\omega(a^2 + \nu^2)S_2 - 2a\omega(S_3 + \nu^2 S_1)} \tag{44}$$

We substitute the expressions for $\sin\omega(\theta_1 - \theta_0)$ and $\cos\omega(\theta_1 - \theta_0)$ from Eq. (44) into Eqs. (42) and (43) , then solve for c_1 and c_2 from Eq. (38). This provides the constants θ_0, c_1, and c_2 in terms of the initial conditions.

In the limit as $\alpha \to 0$ it can be shown with little work that these formulas reduce to the corresponding expressions for the two-body problem without drag. In fact in this limit Eq. (29) reduces to the respective semiparameter and eccentricity of the classical two-body problem. If $c_2 > 0$ then θ_0 defines an apogee, whereas, if $c_2 < 0$ then θ_0 defines a perigee. It is seen from Eqs. (38), (41), and (43) that in the limit as $\alpha \to 0$,

$$c_2 \to \frac{-v_1}{h\sin(\theta_1 - \theta_0)} \tag{45}$$

so that for small values of α, the initial radial velocity v_1 determines which of the three forms Eqs. (30–32) that we get. As is evident from Eq. (45) the formulas Eqs. (42) and (43) produce singularities at $\theta_1 = \theta_0$ in the limit as $\alpha \to 0$.

III. Applications

We now come to the practical problem of calculating the parameters k_1 and k_2 associated with the atmospheric-density representation Eq. (12) to fit the atmospheric data Eq. (11). We assume that $R_{i+1} < R_i$, $i = 1, \ldots, k-1$, and that the orbit of the satellite decays from an initial radial position R_1 to a final position R_k.

We address two separate problems. In the first problem the difference $R_1 - R_k$ is small enough that one value of the parameters k_1, k_2 is sufficient to approximate the atmospheric data with high accuracy. In the second problem, the difference $R_1 - R_k$ is large enough that we must repeatedly recalculate the parameters k_1, k_2 to maintain the desired high accuracy.

A. Minimum Square Error

We seek to approximate the data Eq. (11) by selecting the parameters k_1, k_2 in the function Eq. (12) so as to minimize the quantity

$$I(k_1, k_2) = \sum_{i=1}^{k}\left[\frac{k_1}{R_i} + \frac{k_2}{R_i^2} - \rho(R_i)\right]^2 \tag{46}$$

The solution of this problem is straightforward leading to

$$k_1 = \frac{1}{D}\left[\sum_{i=1}^{k}\frac{\rho(R_i)}{R_i}\sum_{i=1}^{k}\frac{1}{R_i^4} - \sum_{i=1}^{k}\frac{\rho(R_i)}{R_i^2}\sum_{i=1}^{k}\frac{1}{R_i^3}\right] \tag{47}$$

$$k_2 = \frac{1}{D}\left[\sum_{i=1}^{k}\frac{\rho(R_i)}{R_i^2}\sum_{i=1}^{k}\frac{1}{R_i^2} - \sum_{i=1}^{k}\frac{\rho(R_i)}{R_i}\sum_{i=1}^{k}\frac{1}{R_i^3}\right] \tag{48}$$

where

$$D = \sum_{i=1}^{k}\frac{1}{R_i^2}\sum_{i=1}^{k}\frac{1}{R_i^4} - \left(\sum_{i=1}^{k}\frac{1}{R_i^3}\right)^2 \tag{49}$$

If we have only two data points $\rho(R_1)$ and $\rho(R_2)$ this reduces to

$$k_1 = \frac{[R_1^2\rho(R_1) - R_2^2\rho(R_2)]}{R_1 - R_2} \tag{50}$$

$$k_2 = R_1 R_2 \frac{[R_2\rho(R_2) - R_1\rho(R_1)]}{R_1 - R_2} \tag{51}$$

Computer simulations using this approach showed very little advantage in taking more than two data points for an exponential atmospheric density. A far better way to reduce the error in the model is found in the second approach.

B. Piecewise Connected Arcs

We attempt to approximate the data Eq. (11) by repeatedly recalculating the parameters k_1 and k_2 as the satellite drops and moves through the points $R_1, \ldots, R_k$. In this approach the arbitrary constants are calculated anew for each interval from R_i to R_{i+1}.

We begin with the initial conditions $\theta(t_1) = \theta_1$, $R(t_1) = R_1$, $\dot{R}(t_1) = v_1$, and $R^2(t_1)\dot{\theta}(t_1) = h$. It is not necessary to know the numerical value of t_1. The parameters k_1 and k_2 are calculated from Eqs. (50) and (51). Successive values also can be calculated a priori. The constants a, ω, and ν follow from Eqs. (20), (21), (23), and (25).

The values of S_1, S_2, and S_3 are obtained from α, k_1, k_2 and the initial data through Eqs. (39–41). The arbitrary constants θ_0, c_1, c_2 arise from Eqs. (42–44) and (38). These determine R through Eqs. (29–32).

The value θ_2 is reached when the value of R in Eqs. (30) and (31) or Eq. (32) reaches R_2. Setting $\theta = \theta_2$ and $u = u(\theta_2)$ in Eq. (13) the left-hand side defines a new value of h (i.e., h_2) at $\theta(t_2) = \theta_2$. Using this new value of h in Eq. (34) at $\theta = \theta_2$ we obtain a new value for $\dot{R}(t_2) = v_2$. A second iteration is then begun, calculating k_1 and k_2 from R_2 and R_3, and repeating the calculation for a, ω, and ν; S_1, S_2, and S_3; θ_0, c_1, and c_2 from the new data. This process is repeated until the final value of R equals R_k.

C. Model Validation

To examine the accuracy of our formulas, we first compared our density model Eq. (12) over 18 km with the exponential atmospheric model Eq. (7). This comparison was carried with two and six equally spaced data points. At the data points the two models were constrained to coincide with each other . The results of these comparisons are presented in Figs. 1 and 2. We see from these figures that with two data points the maximum relative density error is 0.5%. On the other hand, with six data points the maximum relative error is well under 0.03%.

We then simulated the equations of motion Eqs. (2) and (3) using the (adaptive) Runge–Kutta with relative error of 10^{-8} in each step and $\alpha = 1 \times 10^{-9}$ using the exponential atmospheric model and our density model with six data points. With either model the resulting orbit drop over 160 rad was approximately 18 km. The trajectory deviation between the two models (with a maximum of less than 3 m) is presented in Fig. 3. A comparison of the same numerical solution and exponential atmosphere with the analytic solution given by Eq. (30) is presented in Fig. 4. In this case, the deviation between the analytical and numerical trajectories can reach 35 m. The drag was unrealistically high in this simulation to illustrate the accuracy of our model. Additional simulations were performed for various drag constants. For $\alpha = 2 \times 10^{-9}$ the orbital decay was 40 km over 160 rad, but for brevity, the figures are not included. Calculating the atmospheric-density formulas over 10 equal subintervals of this 40-km drop, the error between the numerical simulation and the analytic formula is less than 190 m over an orbit of 160 rad. Finally, a more realistic simulation was performed in which the orbital decay was 330 m over 160 rad. Although this is yet a very large drag, the error was only 3 m using one interval in the new density model. For brevity the figures are not included.

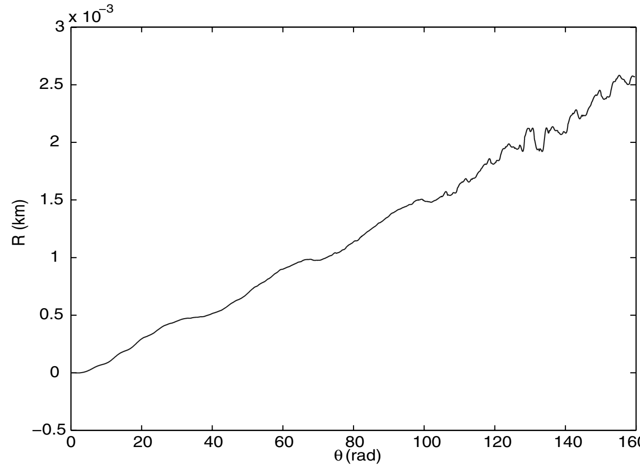

Fig. 3 Difference between the numerical solution of the equations of motion with exponential atmosphere and the analytic solution using the new atmospheric-density model with six data points and $\alpha = 1 \times 10^{-9}$.

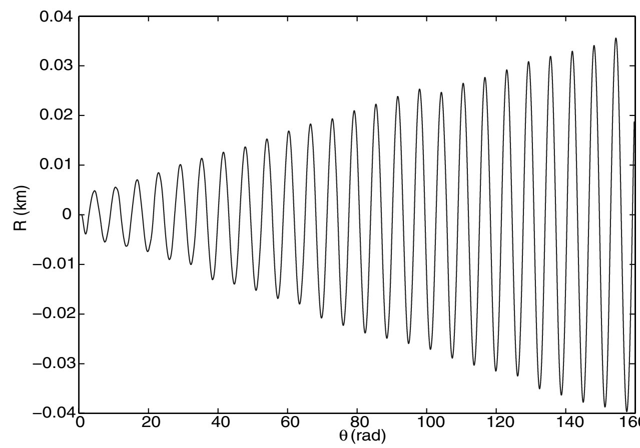

Fig. 4 Difference between the numerical solution of the equations of motion with exponential atmosphere and the analytic solution using the new atmospheric-density model with six data points and $\alpha = 1 \times 10^{-9}$.

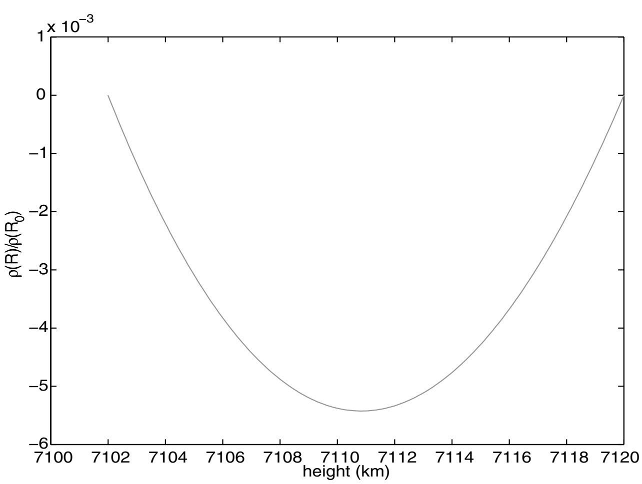

Fig. 1 Difference between the exponential model for the atmospheric density and the new model over 18 km with two data points. The density in both models was normalized to 1 at 7120 km.

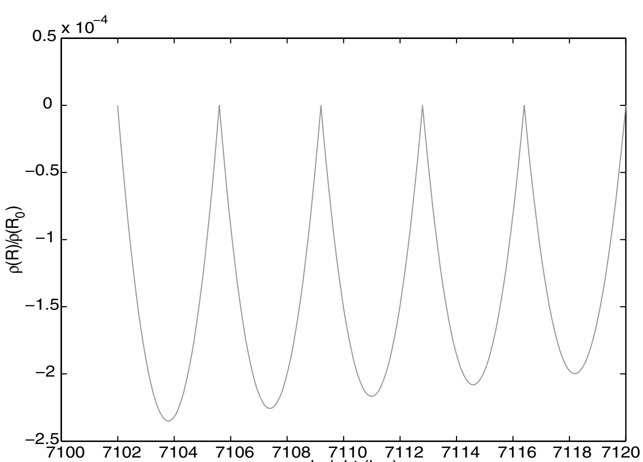

Fig. 2 Same as Fig. 1 but with six equispaced data points.

IV. Flight Time

It follows from Eqs. (4) and (13) that

$$R^2 \dot{\theta} = h e^{-\alpha[k_1(\theta-\theta_1)+k_2 u(\theta)]} \tag{52}$$

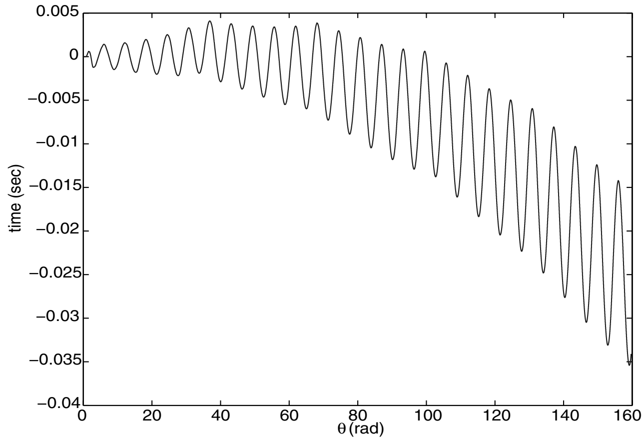

Fig. 5 Time in flight: Difference between the numerical solution with exponential atmosphere and the closed-form expression evaluated numerically for the analytic solution using the new atmospheric-density model with six data points and $\alpha = 1 \times 10^{-9}$.

Separating variables, we can represent the flight time as an integral

$$t - t_1 = \frac{1}{h}\int_{\theta_1}^{\theta} R^2 e^{\alpha[k_1(s-\theta_1)+k_2 u(s)]}\,\mathrm{d}s \tag{53}$$

where $R(\theta)$ is defined by Eqs. (30) and (31) or Eq. (32) and $u(\theta)$ by Eq. (24). Numerical calculations of this integral show that the difference between the values for the time in flight obtained using this formula and those obtained from direct integration of the equations of motion are insignificant (see Fig. 5).

Recalling that Eqs. (29–32) were derived under the assumption that $\alpha \ll 1$, it is possible to find approximate analytical expressions for the flight time. To derive these expressions we observe that for a simulation with one segment over 18 km and $\alpha = 1 \times 10^{-9}$, J changes by no more than one-eighth percent over 160 rad (see Fig. 6). Hence we can approximate J by a constant over this interval of θ. Using the definition of $J = R^2\dot{\theta}$ we therefore have

$$t - t_1 = \frac{1}{J_a}\int_{\theta_1}^{\theta} R^2(s)\,\mathrm{d}s \tag{54}$$

where J_a is an average value of J. To derive a proper analytic expression for this integral we note that for the simulation mentioned above we obtain the following values for the parameters of the trajectory

$$\begin{aligned} a &= 0.0012, \qquad k_1/k_2 = 1.39 \times 10^{-4} \\ ac_1 &= 1.61 \times 10^{-6}, \qquad \nu c_2 = -2 \times 10^{-9} \end{aligned} \tag{55}$$

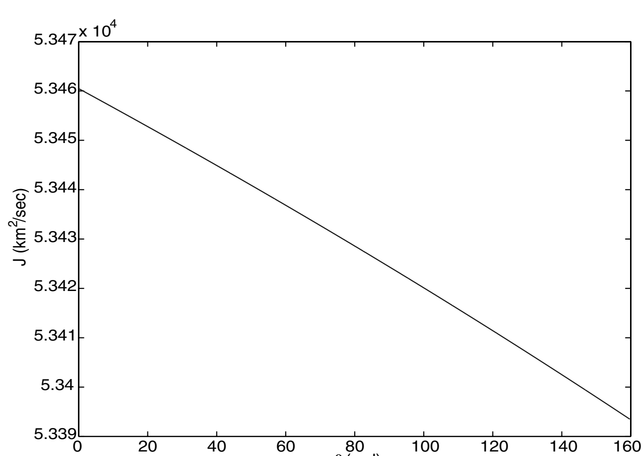

Fig. 6 Specific angular momentum J as a function of θ for $\alpha = 1 \times 10^{-9}$.

From this data and Eq. (33) we infer that it is appropriate to write R^2 in the form

$$R^2 = \frac{k_2^2}{k_1^2\left(1 - \frac{ac_1k_2}{k_1}e^{a\theta} + \frac{\nu c_2 k_2}{k_1}e^{-a\theta/2}\cos\omega(\theta-\theta_0)\right)^2} \tag{56}$$

Hence for the first few revolutions we can use the approximation

$$\frac{1}{(1+x)^2} \approx 1 - 2x$$

to obtain

$$R^2 \approx \frac{k_2^2}{k_1^2}\left[1 + 2\left(\frac{ac_1k_2}{k_1}e^{a\theta} - \frac{\nu c_2k_2}{k_1}e^{-a\theta/2}\cos\omega(\theta-\theta_0)\right)\right] \tag{57}$$

Using this expression for R^2 in Eq. (54) leads to the following explicit formula for the flight time

$$t - t_1 = \frac{G(\theta) - G(\theta_1)}{J_a} \tag{58}$$

where

$$G(\theta) = \frac{k_2^2}{k_1^2}\theta + \frac{2k_2^3}{k_1^3}\{c_1e^{a\theta} + c_2e^{-a\theta/2}\cos[\omega(\theta-\theta_0+\psi)]\} \tag{59}$$

In view of the values of the parameters mentioned above we can approximate Eq. (58) further by the linear expression

$$t - t_1 = \frac{1}{J_a}\left[\frac{k_2^2}{k_1^2}\left(1 + \frac{2ak_2c_1}{k_1}\right)\theta - G(\theta_1)\right] \tag{60}$$

Figure 7 presents the deviation between the flight time obtained using Eq. (58) and those obtained from the numerical solution Eqs. (2) and (3) with exponential atmosphere over 35 rad. In this range the formula is found to be accurate within 1 s. Figure 8 demonstrates the almost linear relationship between the flight time and θ that is obtained using the numerical solution of Eqs. (2) and (3) with exponential atmosphere thus validating Eq. (60).

V. Conclusions

The new formula presented herein approximates atmospheric density as a function of altitude from raw data points, and completes previous work based on other formulas. Closed-form solutions of the orbit equation compared favorably with numerical integration of the equations of motion using data points from an exponential atmospheric-density function. We stress that the exponential

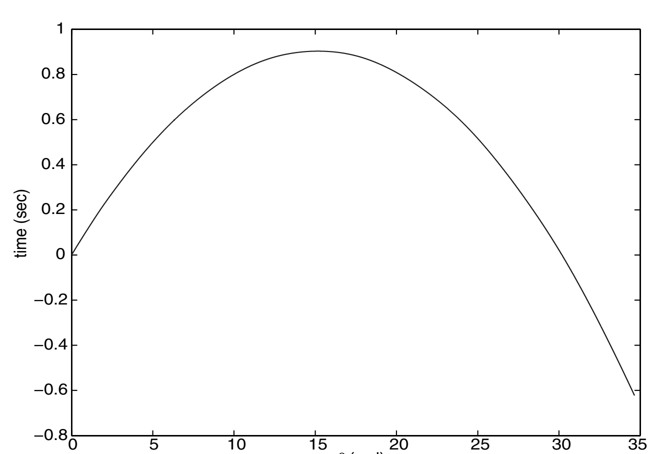

Fig. 7 Deviation between the flight time obtained using Eq. (58) and the one obtained from the numerical solution of the equations of motion with exponential atmosphere for $\alpha = 1 \times 10^{-9}$.

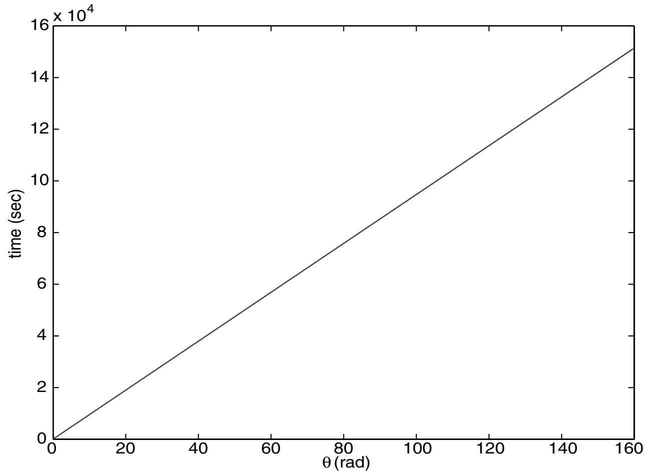

Fig. 8 Flight time as a function of θ from the numerical solution of the equations of motion with exponential atmosphere for $\alpha = 1 \times 10^{-9}$.

function was used only as a convenient test; it is unnecessary to assume an exponentially distributed atmospheric density.

Specifically, these solutions were compared to simulations of an object in near-circular orbit falling from an altitude of 7120–7102 km over some 25 revolutions. Although this drag was unrealistically high, it was useful as a test of the accuracy of the new model. If the approximate atmospheric-density formulas are calculated over five equal subintervals of this 18-km drop, the error from the numerical simulation is less than 35 m over an orbit of 160 rad. Approximate formulas for the time in orbit differed from numerical results by less than 1 s over the first five revolutions. A less unrealistic test having a decay of 330 m over 160 rad produced an error of 3 m using only one interval of the new density model. Additional improvement in accuracy can be obtained by calculation of the parameters of the approximate density formula over smaller subintervals.

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